7.2 Use Similar Triangles to Solve Problems

The geometry of similar figures is a powerful area of mathematics. Similar triangles can be used to measure the heights of objects that are difficult to get to, such as trees, tall buildings, and cliffs. They can also be used to measure distances across rivers and even galaxies!

The students in the photo are using a metre stick and shadows to measure the height of the tree. How can they do this? What role do similar triangles play in this type of problem?

Investigate

Tools
- ruler
- metre stick

How can you apply the properties of similar triangles to solve problems?

Suppose you have a metre stick and it is sunny outside. Your task is to plan a problem solving strategy so that you can determine the height of an inaccessible object, such as a tree, a building, or a cliff.

1. Look at the illustration. Discuss with a partner, or in a small group, how the students could find the height of the tree.

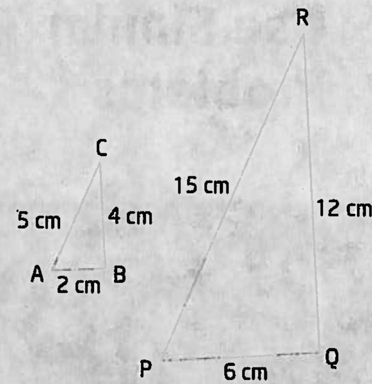
2. Draw a diagram that relates to this problem. Explain how similar triangles are involved.

3. Create some numbers to represent reasonable measures to solve this problem and solve it. Does your answer seem reasonable? Explain.

4. Reflect Trade strategies with another pair or group. Compare strategies. Do you think they will work? Make any improvements you like to your own strategy. Later you will apply your method to solve a real measurement problem.

The **scale factor, *k***, is a useful quantity when working with similar triangles such as the ones shown.

The value of k relating corresponding sides in these two triangles is 3, because if you multiply each side length in $\triangle ABC$ by 3, you obtain the corresponding side length in $\triangle PQR$.



scale factor, *k*

- factor that relates corresponding side lengths of two similar triangles

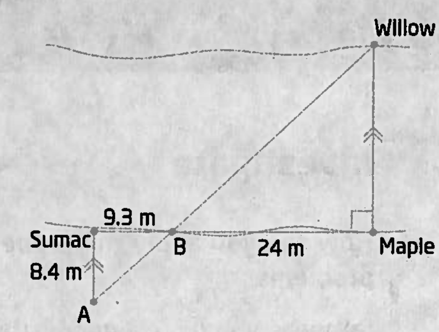
$\triangle ABC$ Side Lengths (cm)	Multiply by k	$\triangle PQR$ Side Lengths (cm)
AB = 2	2 × 3 = 6	PQ = 6
BC = 4	4 × 3 = 12	QR = 12
CA = 5	5 × 3 = 15	RP = 15

You can apply the scale factor to find an unknown side length in one triangle if you know the corresponding side length in a similar triangle.

Example 1 Solve for an Unknown Side

To determine the width of a river, Naomi finds a willow tree and a maple tree that are directly across from each other on opposite shores. Using a third tree on the shoreline, Naomi plants two stakes, A and B, and measures the distances shown.

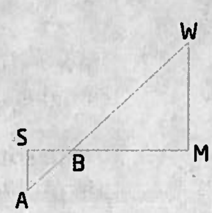
Find the width of the river using the information that Naomi found.



Solution

You can find two similar triangles and then use the scale factor to find the width.

Step 1: Show that $\triangle ABS$ is similar to $\triangle WBM$.

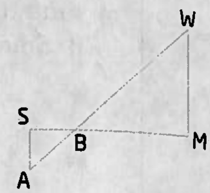


Statement	**Reason**
$\angle ABS = \angle WBM$	These are opposite angles.
$\angle BSA = \angle BMW$	These are both 90°, because AS is parallel to WM.
$\triangle ABS \sim \triangle WBM$	Two pairs of corresponding angles are equal.

If two pairs of corresponding angles are equal, I know that the angles in the third pair are also equal.

Step 2: Find the scale factor.

In $\triangle ABS$ and $\triangle WBM$, BS and BM are corresponding sides. Their ratio gives the scale factor, k:

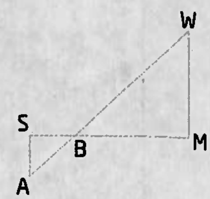


$$k = \frac{BM}{BS}$$
$$= \frac{24}{9.3}$$

Each side in the larger triangle is $\frac{24}{9.3}$ times as long as its corresponding side in the smaller triangle. Leave the scale factor in this form for more accuracy in later calculations.

Step 3: Find the width of the river.

In $\triangle ABS$ and $\triangle WBM$, AS and WM are corresponding sides. Use the scale factor to find WM.



$$\frac{WM}{AS} = k$$
$$\frac{WM}{8.4} = \frac{24}{9.3}$$
$$WM = \frac{24}{9.3}(8.4) \quad \text{Multiply both sides by 8.4.}$$
$$\doteq 21.667$$

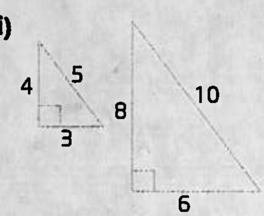
21.677 m is too precise an answer based on the measurements given. One of the measures, BM, is only accurate to the nearest metre. So, it is reasonable to round the answer to the nearest metre.

Therefore, the width of the river is approximately 22 m.

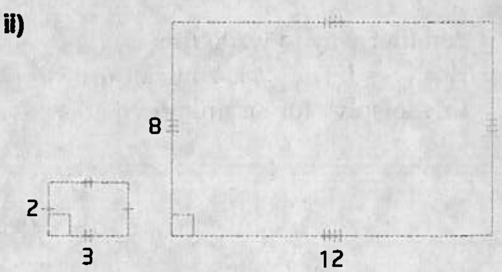
Example 2 Areas of Similar Figures

a) What is the relationship between the areas in each pair of similar figures?

i)



ii)



b) Find the scale factor, k, for each pair of figures.

c) Compare your answers to parts a) and b).

Solution

a) Find the areas of the figures.

i) Smaller Triangle

$$A = \frac{1}{2}bh$$
$$= \frac{1}{2}(3)(4)$$
$$= 6$$

The area of the smaller triangle is 6 square units.

Larger Triangle

$$A = \frac{1}{2}bh$$
$$= \frac{1}{2}(6)(8)$$
$$= 24$$

The area of the larger triangle is 24 square units.

The area of the larger triangle is 4 times the area of the smaller triangle.

ii) Smaller Rectangle

$$A = l \times w$$
$$= 2 \times 3$$
$$= 6$$

The area of the smaller rectangle is 6 square units.

Larger Rectangle

$$A = l \times w$$
$$= 8 \times 12$$
$$= 96$$

The area of the larger rectangle is 96 square units.

The area of the larger rectangle is 16 times the area of the smaller rectangle.

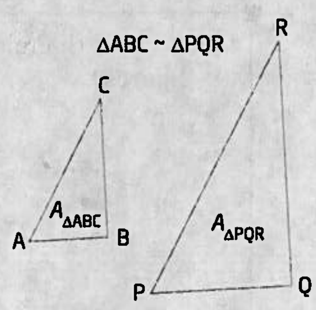
b) i) Since each side length of the larger triangle is 2 times the length of the corresponding side of the smaller triangle, the scale factor is $k = 2$.

ii) Since each side length of the larger rectangle is 4 times the length of the corresponding side of the smaller rectangle, the scale factor is $k = 4$.

c) In both cases, the ratio of the area of the larger figure to the area of the smaller figure is equal to the square of the scale factor, k.

This relationship holds for all similar figures: the ratio of the areas of two similar figures is equal to the square of the scale factor.

Another way to write this is $A_{\triangle PQR} = k^2(A_{\triangle ABC})$. You can use this to solve for an unknown area.

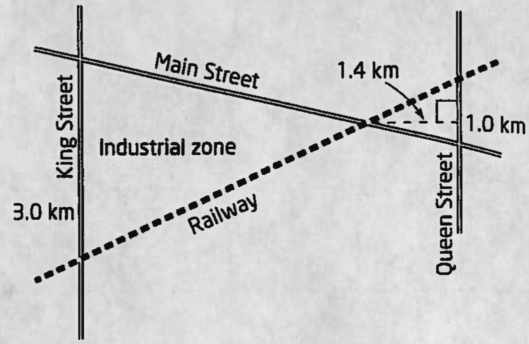


$$\frac{A_{\triangle PQR}}{A_{\triangle ABC}} = k^2$$

Example 3 Solve for an Unknown Area

The shaded area is to be an industrial zone.

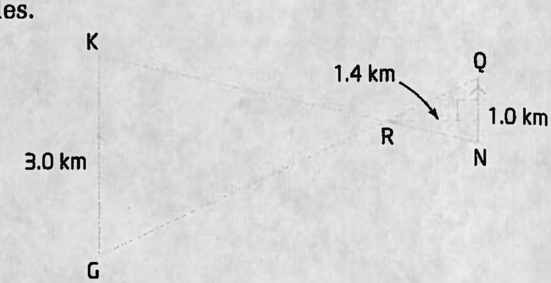
Find the area of the industrial zone. Assume that King and Queen are parallel and that all streets and the track are straight.



Solution

Identify two similar triangles. Find the scale factor and use it to find the area of the larger triangle.

I'll make a simplified diagram.



Statement	Reason
$\angle KRG = \angle NRQ$	Opposite angles are equal.
$\angle RKG = \angle RNQ$	Alternate angles are equal.
$\triangle KRG \sim \triangle NRQ$	Corresponding angles are equal.

If two pairs of corresponding angles are equal, I know that the angles in the third pair are equal, too.

The scale factor is equal to the ratio of corresponding sides:

$$k = \frac{KG}{NQ}$$
$$= \frac{3.0}{1.0}$$
$$= 3$$

Find the area of the smaller triangle using the given information.

$$A_{\triangle NRQ} = \frac{1}{2}bh$$

Apply the formula for the area of a triangle.

$$= \frac{1}{2}(1.0)(1.4)$$
$$= 0.7$$

The area of the smaller triangle is 0.7 km^2. Use this and the scale factor, $k = 3$, to find the area of the larger similar triangle.

$$A_{\triangle KRG} = k^2(A_{\triangle NRQ})$$
$$= 3^2(0.7)$$
$$= 9(0.7)$$
$$= 6.3$$

The area of the industrial zone is 6.3 km^2.